CHAPTER 5
CONSTITUTIVE MODELS

By Brian Moran
Northwestern University
Department of Civil Engineering
Evanston, IL 60208
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In the mathematical description of material behavior, the response of the material is characterized by a constitutive equation which gives the stress as a function of the deformation history of the body. Different constitutive relations allow us to distinguish between a viscous fluid and rubber or concrete, for example. In one-dimensional applications in solid mechanics, the constitutive relation is often referred to as the stress-strain law for the material. In this chapter, some of the most common constitutive models used in solid mechanics applications are described. Constitutive equations for different classes of materials are first presented for the one-dimensional case and are then generalized to multiaxial stress states. Special emphasis is placed on the elastic-plastic constitutive equations for both small and large strains. Some fundamental properties such as reversibility, stability and smoothness are also discussed. An extensive body of theory exists on the thermodynamic foundations of constitutive equations at finite strains and the interested reader is referred to Noll (1973), Truesdell and Noll (1965) and Truesdell (1969). In the present discussion, emphasis is on the mechanical response, although coupling to energy equations and thermal effects are considered.

The implementation of the constitutive relation in a finite element code requires a procedure for the evaluation of the stress given the deformation (or an increment of deformation from a previous state). This may be a straightforward function evaluation as in hyperelasticity or it may require the integration of the rate or incremental form of the constitutive equations. The algorithm for the integration of the rate form of the constitutive relation is called a stress update algorithm. Several stress update algorithms are presented and discussed along with their numerical accuracy and stability. The concept of stress rates arises naturally in the specification of the incremental or rate forms of constitutive equations and this lays the framework for the discussion of linearization of the governing equations in Chapter 6.

In the following Section, the tensile test is introduced and discussed and used to motivate different classes of material behavior. One-dimensional constitutive relations for elastic materials are then discussed in detail in Section 5.2. The special and practically important case of linear elasticity is considered in Section 5.3. In this section, the constitutive relation for general anisotropic linear elasticity is developed. The case of linear isotropic elasticity is obtained by taking account of material symmetry. It is also shown how the isotropic linear elastic constitutive relation may be developed by a generalization of the one-dimensional behavior observed in a tensile test.

Multiaxial constitutive equations for large deformation elasticity are given in Section 5.4. The special cases of hypoelasticity (which often plays an important role in large deformation elastic-plastic constitutive relations) and hyperelasticity are considered. Well-known constitutive models such as Neo-Hookean, Saint Venant Kirchhoff and Mooney-Rivlin material models are given as examples of hyperelastic constitutive relations.
In Section 5.5, constitutive relations for elastic-plastic material behavior for multiaxial stress states for both rate-independent and rate-dependent materials are presented for the case of small deformations. The commonly used von Mises $J_2$-flow theory plasticity models (representative of the behavior of metals) for rate-independent and rate-dependent plastic deformation and the Mohr-Coulomb relation (for the deformation of soils and rock) are presented. The constitutive behavior of elastic-plastic materials undergoing large deformations is presented in Section 5.6.

Well-established extensions of $J_2$-flow theory constitute equations to finite strain resulting in hypoelastic-plastic constitutive relations are discussed in detail in Section 5.7. The Gurson constitutive model which accounts for void-growth and coalescence is given as an illustration of a constitutive relation for modeling material deformation together with damage and failure. The constitutive modeling of single crystals (metal) is presented as an illustration of a set of micromechanically motivated constitutive equation which has proven very successful in capturing the essential features of the mechanical response of metal single crystals. Single crystal plasticity models have also provided a basis for large deformation constitutive models for polycrystalline metals and for other classes of material undergoing large deformation. Hyperelastic-plastic constitutive equations are also considered. In these models, the elastic response is modeled as hyperelastic (rather than hypoelastic) as a means of circumventing some of the difficulties associated with rotations in problems involving geometric nonlinearity.

Constitutive models for the viscoelastic response of polymeric materials are described in section 5.8. Straightforward generalizations of one-dimensional viscoelastic models to multiaxial stress states are presented for the cases of small and large deformations.

Stress update algorithms for the integration of constitutive relations are presented in section 5.9. The radial return and associated so-called return-mapping algorithms for rate-independent materials are presented first. Stress-update schemes for rate dependent material are then presented and the concept of algorithmic tangent modulus is introduced. Issues of accuracy and stability of the various schemes are introduced and discussed.

5.1. The Stress-Strain Curve

The relationship between stress and deformation is represented by a constitutive equation. In a displacement based finite element formulation, the constitutive relation is used to represent stress or stress increments in terms of displacement or displacement increments respectively. Consequently, a constitutive equation for general states of stress and stress and deformation histories is required for the material. The purpose of this chapter is to present the theory and development of constitutive equations for the most commonly observed classes of material behavior. To the product designer or analyst, the choice of material model is very important and may not always be obvious. Often the only information available is general knowledge and experience about the material behavior along with perhaps a few stress strain curves. It is the analyst's task to choose the appropriate constitutive model from available libraries in the finite element code or to develop a user supplied constitutive routine if no suitable constitutive equation is available. It is important for the engineer to understand what the key features of the constitutive model for the material are, what assumptions have gone into the development of the model, how suitable the model is for the material in question, how appropriate the model is for the expected load and deformation regime and what numerical issues are involved in the implementation of the model to assure accuracy and stability of the numerical procedure. As will be seen below, the analyst needs to have a broad understanding of relevant areas of mechanics of materials, continuum mechanics and numerical methods.
Many of the essential features of the stress-strain behavior of a material can be obtained from a set of stress-strain curves for the material response in a state of one-dimensional stress. Both the physical and mathematical descriptions of the material behavior are often easier to describe for one-dimensional stress states than for any other. Also, as mentioned above, often the only quantitative information the analyst has about the material is a set of stress-strain curves. It is essential for the analyst to know how to characterize the material behavior on the basis of such stress-strain curves and to know what additional tests, if any, are required so that a judicious choice of constitutive equation can be made. For these reasons, we begin our treatment of constitutive models and their implementation in finite element codes with a discussion of the tensile test. As will be seen, constitutive equations for multiaxial states are often based on simple generalizations of the one-dimensional behavior observed in tensile tests.

5.1.1. The Tensile Test

The stress-strain behavior of a material in a state of uniaxial (one-dimensional) stress can be obtained by performing a tensile test (Figure 5.1). In the tensile test, a specimen is gripped at each end in a testing machine and elongated at a prescribed rate. The elongation \( \delta \) of the gage section and the force \( T \) required to produce the elongation are measured. A plot of load versus elongation (for a typical metal) is shown in Figure 5.1. This plot represents the response of the specimen as a structure. In order to extract meaningful information about the material behavior from this plot, the contributions of the specimen geometry must be removed. To do this, we plot load per unit area (or stress) of the gage cross-section versus elongation per unit length (or strain). Even at this stage, decisions need to be made: Do we use the original area and length or the instantaneous ones? Another way of stating this question is what stress and strain measures should we use? If the deformations are sufficiently small that distinctions between original and current geometries are negligible for the purposes of computing stress and strain, a small strain theory is used and a small strain constitutive relation developed. Otherwise, full nonlinear kinematics are used and a large strain (or finite deformation) constitutive relation is developed. From Chapter 3 (Box 3.2), it can be seen that we can always transform from one stress or strain measure to another but it is important to know precisely how the original stress-strain relation is specified. A typical procedure is as follows:

Define the stretch \( \lambda = L/L_0 \) where \( L = L_0 + \delta \) is the length of the gage section associated with elongation \( \delta \). Note that \( \lambda = \sqrt[1]{F} \) where \( F \) is the deformation gradient. The nominal (or engineering stress) is given by

\[
P = \frac{T}{A_0} \tag{5.1.1}
\]

where \( A_0 \) is the original cross-sectional area. The engineering strain is given by

\[
\varepsilon = \frac{\delta}{L_0} = \lambda - 1 \tag{5.1.2}
\]

A plot of engineering stress versus engineering strain for a typical metal is given in Figure 5.2.

Alternatively, the stress-strain response may be given in terms of true stress. The Cauchy (or true) stress is given by
\[
\sigma = \frac{T}{A}
\]  \hspace{1cm} (5.1.3)

where \(A\) is the current (instantaneous) area of the cross-section. A measure of true strain is derived by considering an increment of true strain as change in length per unit current length, i.e., \(d\epsilon_{\text{true}} = dL/L\). Integrating this relation from the initial length \(L_0\) to the current length \(L\) gives

\[
\epsilon_{\text{true}} = \int_{L_0}^{L} \frac{dL}{L} = \ln \left( \frac{L}{L_0} \right) = \ln \lambda
\]  \hspace{1cm} (5.1.4)

Taking the material time derivative of this expression gives

\[
\dot{\epsilon}_{\text{true}} = \frac{\dot{\lambda}}{\lambda} = D_{11}
\]  \hspace{1cm} (5.1.5)

i.e., in the one-dimensional case, the time-derivative of the true strain is equal to the rate of deformation given by Eq. (3.3.19). This is not true in general unless the principal axes of the deformation are fixed.

To plot true stress versus true strain, we need to know the cross-sectional area \(A\) as a function of the deformation and this can be measured during the test. If the material is incompressible, then the volume remains constant and we have \(A_0L_0 = AL\) which can be written as

\[
A = A_0/\lambda
\]  \hspace{1cm} (5.1.6)

and therefore the Cauchy stress is given by

\[
\sigma = \frac{T}{A} = \frac{\lambda T}{A_0} = \lambda P
\]  \hspace{1cm} (5.1.7)

A plot of true stress versus true strain is given in Figure 5.3.

The nominal or engineering stress is written in tensorial form as \(P = P_{11}e_1 \otimes e_1\) where \(P_{11} = P = T/A_0\). From Box 3.2, the Cauchy (or true) stress is given by

\[
\sigma = J^{-1}F^T \cdot P
\]  \hspace{1cm} (5.1.8)

where \(J = \det F\) and it follows that

\[
\sigma = \sigma_{11} e_1 \otimes e_1 = J^{-1} \lambda P_{11} e_1 \otimes e_1
\]  \hspace{1cm} (5.1.9)

For the special case of an incompressible material \(J = 1\) and Eq. (5.1.9) is equivalent to Eq. (5.1.7).
Prior to the development of instabilities (such as the well known phenomenon of necking) the deformation in the gage section of the bar can be taken to be homogeneous. The deformation gradient, Eq. (3.2.14), is written as

\[ F = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \]  

(5.1.10)

where \( \lambda_1 = \lambda \) is the stretch in the axial direction (taken to be aligned with the \( x_1 \)-axis of a rectangular Cartesian coordinate system) and \( \lambda_2 = \lambda_3 \) are the stretches in the lateral directions. For an incompressible material \( J = \det F = \lambda_1 \lambda_2 \lambda_3 = 1 \) and thus \( \lambda_2 = \lambda_3 = \lambda^{-2} \).

Now assume that we can represent the relationship between nominal stress and engineering strain in the form of a function

\[ P_{11} = s_0(\varepsilon_{11}) \]  

(5.1.11)

where \( \varepsilon_{11} = \lambda - 1 \) is the engineering strain. We can regard (5.1.11) as a stress-strain equation for the material undergoing uniaxial stressing at a given rate of deformation. At this stage we have not introduced unloading or made any assumptions about the material response. From equation (5.1.9), the true stress (for an incompressible material) can be written as

\[ \sigma_{11} = \lambda s_0(\varepsilon_{11}) = s(\lambda) \]  

(5.1.12)

where the relation between the functions is \( s(\lambda) = \lambda s_0(\lambda - 1) \). This is an illustration of how we obtain different functional representations of the constitutive relation for the same material depending on what measures of stress and deformation are used. It is especially important to keep this in mind when dealing with multiaxial constitutive relations at large strains.

A material for which the stress-strain response is independent of the rate of deformation is said to be rate-independent; otherwise it is rate-dependent. In Figures 5.4 a,b, the one-dimensional response of a rate-independent and a rate-dependent material are shown respectively for different nominal strain rates. The nominal strain rate is defined as \( \dot{\varepsilon} = \dot{\delta}/L_0 \). Using the result \( \dot{\delta} = \dot{L} \) and therefore \( \dot{\delta}/L_0 = \dot{L}/L_0 = \dot{\lambda} \) it follows that the nominal strain rate is equivalent to the rate of stretching, i.e., \( \dot{\varepsilon} = \dot{\lambda} = \dot{F}_{11} \). As can be seen, the stress-strain curve for the rate-independent material is independent of the strain rate while for the rate-dependent material the stress strain curve is elevated at higher rates. The elevation of stress at the higher strain rate is the typical behavior observed in most materials (such as metals and polymers). A material for which an increase in strain rate gives rise to a decrease in the stress strain curve is said to exhibit anomalous rate-dependent behavior.

In the description of the tensile test given above no unloading was considered. In Figure 5.5 unloading behaviors for different types of material are illustrated. For elastic materials, the unloading stress strain curve simply retraces the loading one. Upon complete unloading, the material returns to its initial unstretched state. For elastic-plastic materials, however, the unloading curve is different from the loading curve. The slope of the unloading curve is typically that of the elastic (initial) portion of the stress strain curve. This
results in permanent strains upon unloading as shown in Figure 5.5b. Other materials exhibit behaviors between these two extremes. For example, the unloading behavior for a brittle material which develops damage (in the form of microcracks) upon loading exhibits the unloading behavior shown in Figure 5.5c. In this case the elastic strains are recovered when the microcracks close upon removal of the load. The initial slope of the unloading curve gives information about the extent of damage due to microcracking.

In the following section, constitutive relations for one-dimensional linear and nonlinear elasticity are introduced. Multiaxial constitutive relations for elastic materials are discussed in section 9.3 and for elastic-plastic and viscoelastic materials in the remaining sections of the chapter.

5.2. One-Dimensional Elasticity

A fundamental property of elasticity is that the stress depends only on the current level of the strain. This implies that the loading and unloading stress strain curves are identical and that the strains are recovered upon unloading. In this case the strains are said to be reversible. Furthermore, an elastic material is rate-independent (no dependence on strain rate). It follows that, for an elastic material, there is a one-to-one correspondence between stress and strain. (We do not consider a class of nonlinearly elastic materials which exhibit phase transformations and for which the stress strain curve is not one-to-one. For a detailed discussion of the treatment of phase transformations within the framework of nonlinear elasticity see (Knowles, ).)

We focus initially on elastic behavior in the small strain regime. When strains and rotations are small, a small strain theory (kinematics, equations of motion and constitutive equation) is often used. In this case we make no distinction between the various measures of stress and strain. We also confine our attention to a purely mechanical theory in which thermodynamic effects (such as heat conduction) are not considered. For a nonlinear elastic material (small strains) the relation between stress and strain can be written as

\[ \sigma_x = s(\varepsilon_x) \]  

(5.2.1)

where \( \sigma_x \) is the Cauchy stress and \( \varepsilon_x = \delta / L_0 \) is the linear strain, often known as the engineering strain. Here \( s(\varepsilon_x) \) is assumed to be a monotonically increasing function. The assumption that the function \( s(\varepsilon_x) \) is monotonically increasing is crucial to the stability of the material: if at any strain \( \varepsilon_x \), the slope of the stress strain curve is negative, i.e., \( ds / d\varepsilon_x < 0 \) then the material response is unstable. Such behavior can occur in constitutive models for materials which exhibit phase transformations (Knowles). Note that reversibility and path-independence are implied by the structure of (5.2.1): the stress \( \sigma_x \) for any strain \( \varepsilon_x \) is uniquely given by (5.2.1). It does not matter how the strain reaches the value \( \varepsilon_x \). The generalization of (5.2.1) to multiaxial large strains is a formidable mathematical problem which has been addressed by some of the keenest minds in the 20th century and still encompasses open questions (see Ogden, 1984, and references therein). The extension of (5.2.1) to large strain uniaxial behavior is presented later in this Section. Some of the most common multiaxial generalizations to large strain are discussed in Section 5.3.
In a purely mechanical theory, reversibility and path-independence also imply the absence of energy dissipation in deformation. In other words, in an elastic material, deformation is not accompanied by any dissipation of energy and all energy expended in deformation is stored in the body and can be recovered upon unloading. This implies that there exists a potential function \( \rho^{\text{w int}}(\varepsilon_x) \) such that

\[
\sigma_x = s(\varepsilon_x) = \frac{\rho^{\text{w int}}(\varepsilon_x)}{d\varepsilon_x}
\]

(5.2.2)

where \( \rho^{\text{w int}}(\varepsilon_x) \) is the strain energy density per unit volume. From Eq. (5.2.2) it follows that

\[
\rho^{\text{w int}}(\varepsilon_x) = \sigma_x d\varepsilon_x
\]

(5.2.3)

which when integrated gives

\[
\rho^{\text{w int}} = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x
\]

(5.2.4)

This can also be seen by noting that \( \sigma_x d\varepsilon_x = \sigma_x \dot{\varepsilon}_x dt \) is the one-dimensional form of \( \sigma_{ij} D_{ij} dt \) for small strains.

One of the most obvious characteristics of a stress-strain curve is the degree of nonlinearity it exhibits. For many materials, the stress strain curve consists of an initial linear portion followed by a nonlinear regime. Also typical is that the material behaves elastically in the initial linear portion. The material behavior in this regime is then said to be linearly elastic. The regime of linear elastic behavior is typically confined to strains of no more than a few percent and consequently, small strain theory is used to describe linear elastic materials or other materials in the linear elastic regime.

For a linear elastic material, the stress strain curve is linear and can be written as

\[
\sigma_x = E\varepsilon_x
\]

(5.2.5)

where the constant of proportionality is Young's modulus, \( E \). This relation is often referred to as Hooke's law. From Eq. (5.2.4) the strain energy density is therefore given by

\[
p^{\text{w int}} = \frac{1}{2} E\varepsilon_x^2
\]

(5.2.6)

which is a quadratic function of strains. To avoid confusion of Young's modulus with the Green strain, note that the Green (Lagrange) strain is always subscripted or in boldface.

Because energy is expended in deforming the body, the strain energy \( w^{\text{int}} \) is assumed to be a convex function of strain, i.e., \( w^{\text{int}}(\varepsilon_x^1) - w^{\text{int}}(\varepsilon_x^2)(\varepsilon_x^1 - \varepsilon_x^2) \geq 0 \), equality
if \( \varepsilon_1 = \varepsilon_2 \). If \( w^{\text{int}} \) is non-convex function, this implies that energy is released by the body as it deforms, which can only occur if a source of energy other than mechanical is present and is converted to mechanical energy. This is the case for materials which exhibit phase transformations. Schematics of convex and non-convex energy functions along with the corresponding stress strain curves given by (5.2.2) are shown in Figure 5.6.

In summary, the one-dimensional behavior of an elastic material is characterized by three properties which are all interrelated

\[ \text{path– independence } \iff \text{ reversible } \iff \text{nondissipative} \]

These properties can be embodied in a material model by modeling the material response by an elastic potential.

The extension of elasticity to large strains in one dimension is rather straightforward: it is only necessary to choose a measure of strain and define an elastic potential for the (work conjugate) stress. Keep in mind that the existence of a potential implies reversibility, path-independence and absence of dissipation in the deformation process. We can choose the Green strain as a measure of strain \( E_X \) and write

\[ S_X = \frac{d\Psi}{dE_X} \tag{5.2.7} \]

The fact that the corresponding stress is the second Piola-Kirchhoff stress follows from the work (power) conjugacy of the second Piola-Kirchhoff stress and the Green strain, i.e., recalling Box 3.4 and, specializing to one dimension, the stress power per unit reference volume is given by \( \dot{\Psi} = S_X \dot{E}_X \).

The potential \( \Psi \) in (5.2.7) reduces to the potential (5.2.2) as the strains become small. Elastic stress-strain relationships in which the stress can be obtained from a potential function of the strains are called hyperelastic.

The simplest hyperelastic relation (for large deformation problems in one dimension) results from a potential which is quadratic in the Green strain:

\[ \Psi = \frac{1}{2} E E_X^2 \tag{5.2.8} \]

Then,

\[ S_X = E E_X \tag{5.2.9} \]

by equation (5.2.7), so the relation between these stress and strain measures is linear. At small strains, the relation reduces to Hooke's Law (5.2.5).

We could also express the elastic potential in terms of any other conjugate stress and strain measures. For example, it was pointed out in Chapter 3 that the quantity \( \bar{U} = U - I \) is a valid strain measure (called the Biot strain), and that in one-dimension the conjugate stress is the nominal stress \( P_X \), so
\[ P_X = \frac{d\Psi}{d\bar{U}_X} = \frac{d\Psi}{dU_X} \]  

(5.2.10)

We can write the second form in (5.2.10) because the unit tensor \( I \) is constant and hence \( d\bar{U}_X = dU_X \). It is interesting to observe that linearity in the relationship between a certain pair of stress and strain measures does not imply linearity in other conjugate pairs. For example if \( S_X = EE_X \) it follows that \( P_X = E(U_X^2 + 2U_X)/2 \).

A material for which the rate of Cauchy stress is related to the rate of deformation is said to be hypoelastic. The relation is generally nonlinear and is given by

\[ \dot{\sigma} = f(\sigma_x, D_x) \]  

(5.2.11)

where a superposed dot denotes the material time derivative and \( D_x \) is the rate of deformation. A particular linear hypoelastic relation is given by

\[ \dot{\sigma}_x = ED_x = E \frac{\dot{\lambda}_x}{\lambda_x} \]  

(5.2.12)

where \( E \) is Young's modulus and \( \lambda_x \) is the stretch. Integrating, this relation we obtain

\[ \sigma_x = E \ln \lambda_x \]  

(5.2.13)

or

\[ \sigma_x = \frac{d}{d\lambda_x} \int_1^{\lambda_x} E \ln \xi d\xi \]  

(5.2.14)

which is a hyperelastic relation and thus path-independent. However, for multiaxial problems, hypoelastic relations can not in general be transformed to hyperelastic. Multiaxial constitutive models for hypoelastic, elastic and hyperelastic materials are described in Sections 5.3 and 5.4 below.

A hypoelastic material is, in general, strictly path-independent only in the one-dimensional case. However, if the elastic strains are small, the behavior is close enough to path-independent to model elastic behavior. Because of the simplicity of hypoelastic laws, a muti-axial generalization of (5.2.11) is often used in finite element software to model the elastic response of materials in large strain elastic-plastic problems (see Section 5.7 below).

For the case of small strains, equation (9.2.12) above can be written as

\[ \dot{\sigma}_x = E\dot{\epsilon}_x \]  

(5.2.15)

which is the rate form (material time derivative) of Hooke's law (5.2.5).
For the general elastic relation (5.2.1) above, the function \( s(\varepsilon_x) \) was assumed to be monotonically increasing. The corresponding strain energy is shown in Figure 5.6b and can be seen to be a convex function of strain. Materials for which \( s(\varepsilon_x) \) first increases and then decreases exhibit *strain-softening* or unstable material response (i.e., \( ds/d\varepsilon_x < 0 \)). A special form of non-monotonic response is illustrated in Figure 5.7a. Here, the function \( s(\varepsilon_x) \) increases monotonically again after the strain-softening stage. The corresponding energy is shown in Figure 5.7b. This type of non-convex strain energy has been used in nonlinear elastic models of phase transformations (Knowles). At a given stress \( \sigma \) below \( \sigma_M \) the material may exist in either of the two strained states \( \varepsilon_a \) or \( \varepsilon_b \) as depicted in the figure. The reader is referred to (Knowles) for further details including such concepts as the energetic force on a phase boundary (interface driving traction) and constitutive relations for interface mobility.

### 5.3. Multiaxial Linear Elasticity

In many engineering applications involving small strains and rotations, the response of the material may be considered to be linearly elastic. The most general way to represent a linear relation between the stress and strain tensors is given by

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \sigma = C : \varepsilon
\]  

(5.3.1)

where \( C_{ijkl} \) are components of the 4th-order tensor of elastic moduli. This represents the generalization of (5.2.5) to multiaxial states of stress and strain and is often referred to as the generalized Hooke's law which incorporates fully anisotropic material response.

The strain energy per unit volume, often called the elastic potential, as given by (5.2.4) is generalized to multiaxial states by:

\[
W = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \varepsilon : C : \varepsilon
\]  

(5.3.2)

The stress is then given by

\[
\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \quad \sigma = \frac{\partial W}{\partial \varepsilon}
\]  

(5.3.3)

which is the tensor equivalent of (5.2.2). The strain energy is assumed to be positive-definite, i.e.,

\[
W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \varepsilon : C : \varepsilon \geq 0
\]  

(5.3.4)

with equality if and only if \( \varepsilon_{ij} = 0 \) which implies that \( C \) is a positive-definite fourth-order tensor. From the symmetries of the stress and strain tensors, the material coefficients have the so-called minor symmetries

\[
C_{ijkl} = C_{jikl} = C_{ijlk}
\]  

(5.3.5)
and from the existence of a strain energy potential (5.3.2) it follows that

$$C_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}, \quad C = \frac{\partial^2 W}{\partial \varepsilon \partial \varepsilon}$$

(5.3.6)

If $W$ is a smooth $\left(C^1\right)$ function of $\varepsilon$, Eq. (5.3.6) implies a property called major symmetry:

$$C_{ijkl} = C_{klij}$$

(5.3.7)

since smoothness implies

$$\frac{\partial^2 W}{\partial \varepsilon_{ij} \varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{kl} \varepsilon_{ij}}$$

(5.3.8)

The general fourth-order tensor $C_{ijkl}$ has $3^4 = 81$ independent constants. These 81 constants may also be interpreted as arising from the necessity to relate 9 components of the complete stress tensor to 9 components of the complete strain tensor, i.e., $81 = 9 \times 9$. The symmetries of the stress and strain tensors require only that 6 independent components of stress be related to 6 independent components of strain. The resulting minor symmetries of the elastic moduli therefore reduce the number of independent constants to $6 \times 6 = 36$. Major symmetry of the moduli, expressed through Eq. (5.3.7) reduces the number of independent elastic constants to $n(n+1)/2 = 21$, for $n = 6$, i.e., the number of independent components of a $6 \times 6$ matrix.

Considerations of material symmetry further reduce the number of independent material constants. This will be discussed below after the introduction of Voigt notation. An isotropic material is one which has no preferred orientations or directions, so that the stress-strain relation is identical when expressed in component form in any rectangular Cartesian coordinate system. The most general constant isotropic fourth-order tensor can be shown to be a linear combination of terms comprised of Kronecker deltas, i.e., for an isotropic linearly elastic material

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

(5.3.9)

Because of the symmetry of the strain and the associated minor symmetry $C_{ijkl} = C_{ijlk}$ it follows that $\mu' = 0$. Thus Eq. (6.3.9) is written

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \quad C = \lambda I \otimes I + 2\mu I$$

(5.3.10)

and the two independent material constants $\lambda$ and $\mu$ are called the Lamé constants.

The stress strain relation for an isotropic linear elastic material may therefore be written as
\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2 \mu \varepsilon_{ij} = C_{ijkl} \varepsilon_{kl}, \quad \sigma = \lambda \text{trace}(\varepsilon) I + 2 \mu \varepsilon \]  

(5.3.11)

**Voigt Notation**

Voigt notation employs the following mapping of indices to represent the components of stress, strain and the elastic moduli in convenient matrix form:

\[
11 \rightarrow 1 \quad 22 \rightarrow 2 \quad 33 \rightarrow 3 \\
23 \rightarrow 4 \quad 13 \rightarrow 5 \quad 12 \rightarrow 6
\]

(5.3.12)

Thus, stress can be written as a column matrix \( \{ \sigma \} \) with

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{22} & \sigma_{23} & \sigma_{33} \\
\sigma_{sym} & \sigma_{23} & \sigma_{33}
\end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
\]

(5.3.13)

or

\[
\{ \sigma \}^T = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6] \\
= [\sigma_1, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}]
\]

(5.3.14)

Strain is similarly written in matrix form with the exception that a factor of 2 is introduced on the shear terms, i.e.,

\[
\{ \varepsilon \}^T = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6] \\
= [\varepsilon_1, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}]
\]

(5.3.15)

The factor of 2 is included in the shear strain terms to render the stress and strain column matrices work conjugates, i.e.,

\[
W = \frac{1}{2} \sigma^T \varepsilon = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \sigma : \varepsilon
\]

(5.3.16)

The matrix of elastic constants is obtained from the tensor components by mapping the first and second pairs of indices according to (5.3.12). For example, \( C_{11} = C_{1111} \), \( C_{12} = C_{1122} \), \( C_{14} = C_{1123} C_{56} = C_{1312} \) etc. For example, the stress strain relation for \( \sigma_{11} \) is given by
\[ \sigma_{11} = C_{1111} \varepsilon_{11} + C_{1112} \varepsilon_{12} + C_{1113} \varepsilon_{13} \\
+ C_{1121} \varepsilon_{21} + C_{1122} \varepsilon_{22} + C_{1123} \varepsilon_{23} \\
+ C_{1131} \varepsilon_{31} + C_{1132} \varepsilon_{32} + C_{1133} \varepsilon_{33} \\
= C_{11} \varepsilon_{1} + \frac{1}{2} C_{16} \varepsilon_{6} + \frac{1}{2} C_{15} \varepsilon_{5} + \frac{1}{2} C_{16} \varepsilon_{6} \\
+ \frac{1}{2} C_{12} \varepsilon_{2} + \frac{1}{2} C_{14} \varepsilon_{4} + \frac{1}{2} C_{15} \varepsilon_{5} + \frac{1}{2} C_{14} \varepsilon_{4} + \frac{1}{2} C_{13} \varepsilon_{3} \\
= C_{11} \varepsilon_{1} + C_{12} \varepsilon_{2} + C_{13} \varepsilon_{3} + C_{14} \varepsilon_{4} + C_{15} \varepsilon_{5} + C_{16} \varepsilon_{6} \\
= C \varepsilon_{1} \]  

(5.3.17)

and similarly for the remaining components of stress. The constitutive relation may then be written in matrix form as

\[ \sigma = C \varepsilon, \quad \sigma_i = C_{ij} \varepsilon_j \]  

(5.3.18)

Major symmetry (5.3.7) implies that the matrix \([C]\), of elastic constants is symmetric with 21 independent entries, i.e.,

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} = 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
\]  

(5.3.19)

The relation (5.3.19) holds for arbitrary anisotropic linearly elastic materials. The number of independent material constants is further reduced by considerations of material symmetry (see Nye (1985) for example). For example, if the material has a plane of symmetry, say the \(x_1\)-plane, the elastic moduli must remain unchanged when the coordinate system is changed to one in which the \(x_1\)-axis is reflected through the \(x_1\)-plane. Such a reflection introduces a factor of -1 for each term in the moduli \(C_{ijkl}\) in which the index 1 appears. Because the \(x_1\) plane is a plane of symmetry, the moduli must remain unchanged under this reflection and therefore any term in which the index 1 appears an odd number of times must vanish. This occurs for the terms \(C_{\alpha 5}\) and \(C_{\alpha 6}\) for \(\alpha = 1, 2, 3\). For an orthotropic material (e.g., wood or aligned fiber reinforced composites) for which there are three mutually orthogonal planes of symmetry, this procedure can be repeated for all three planes to show that there are only 9 independent elastic constants and the constitutive matrix is written as
An isotropic material is one for which there are no preferred orientations. Recall that an isotropic tensor is one which has the same components in any (rectangular Cartesian) coordinate system. Many materials (such as metals and ceramics) can be modeled as isotropic in the linear elastic range and the linear isotropic elastic constitutive relation is perhaps the most widely used material model in solid mechanics. There are many excellent treatises on the theroy of elasticity and the reader is referred to (Timoshenko and Goodier, 1975; Love, and Green and Zerna,) for more a more detailed description than that given here. As in equation (5.3.10) above the number of independent elastic constants for an isotropic linearly elastic material reduces to 2. The isotropic linear elastic law is written in Voigt notation as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 & 0 & 0 \\
C_{44} & 0 & 0 & 0 & 0 & 0 \\
sym & C_{55} & 0 & 0 & 0 & 0 \\
& & & & & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
\]

(5.3.20)

where \( \lambda \) and \( \mu \$ are the Lamé constants.

The isotropic linear elastic relation (5.3.21) has been derived from the general anisotropic material model (5.3.19) by considering material symmetry. It is instructive to see also how the relation (5.3.21) may be generalized from the particular by starting with the case of a linearly elastic isotropic bar under uniaxial stress. For small strains, the axial strain in the bar is given by the elongation per unit original length, i.e., \( \varepsilon_{11} = \delta/L_0 \) and from Hooke's law (5.2.5)

\[
\varepsilon_{11} = \frac{\sigma_{11}}{E}
\]

(5.3.22)

The lateral strain in the bar is given by \( \varepsilon_{22} = \varepsilon_{33} = \Delta D/D_0 \) where \( \Delta D \) is the change in the original diameter \( D_0 \). For an isotropic material, the lateral strain is related to the axial strain by

\[
\varepsilon_{22} = \varepsilon_{33} = -v\varepsilon_{11} = -v\frac{\sigma_{11}}{E}
\]

(5.3.23)
where \( v \) is Poisson’s ratio. To generalize these relations to multiaxial stress states, consider the stress state shown in Figure 5.8 where the primed coordinate axes are aligned with the directions of principal stress. Because of the linearity of the material response, the strains due to the individual stresses may be superposed to give

\[
\begin{align*}
\varepsilon'_{11} &= \frac{\sigma'_{11}}{E} - v(\sigma'_{22} + \sigma'_{33}) \\
\varepsilon'_{22} &= \frac{\sigma'_{22}}{E} - v(\sigma'_{11} + \sigma'_{33}) \\
\varepsilon'_{33} &= \frac{\sigma'_{33}}{E} - v(\sigma'_{11} + \sigma'_{22}) 
\end{align*}
\]  

(5.3.24)

Referring the stresses and strains to an arbitrary set of (rectangular Cartesian) axes by using the relation (3.2.30) for transformation of tensor components gives

\[
\varepsilon_{ij} = \frac{(1 + v)}{E} \sigma_{ij} - \frac{v}{E} \sigma_{kk} \delta_{ij}
\]

(5.3.25)

**Exercise 5.1.** Derive Eq. (5.3.25) from (5.3.24) and (3.2.30).

The relation between shear stress and shear strain is given by (for example) \( \sigma_{12} = 2\mu\varepsilon_{12} \) where the shear modulus (or modulus of rigidity) \( \mu \) is defined as

\[
\mu = \frac{E}{2(1 + v)}
\]

(E5.1.1)

From Eq.(5.3.25) it follows that

\[
\varepsilon_{kk} = \frac{(1 - 2v)}{E} \sigma_{kk} = \frac{\sigma_{kk}}{3K}
\]

(E5.1.2)

where

\[
K = \frac{E}{3(1-2v)} = \lambda + \frac{2\mu}{3}
\]

(E5.1.3)

is the bulk modulus. Introducing the Lamé constant \( \lambda \), given by

\[
\lambda = \frac{vE}{(1 + v)(1 - 2v)}
\]

(E5.1.4)

the bulk modulus is written as

\[
K = \lambda + \frac{2\mu}{3}
\]

(E5.1.5)
From (5.3.29) and (5.3.26), the quantity we obtain the relation \( \nu/E = \lambda/2\mu \). Using this result and (5.3.26) in (5.3.25), the stress strain relation is given by

\[
\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{kk} \delta_{ij}
\]

(E5.1.6)

Using (5.3.27) this expression may be inverted to give Eq.(5.3.11), the generalized Hooke’s law.

Writing the stress and strain tensors as the sum of deviatoric and hydrostatic or volumetric parts, i.e.,

\[
\sigma_{ij} = s_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij}
\]

\[
\varepsilon_{ij} = e_{ij} + \frac{1}{3} \varepsilon_{mm} \delta_{ij}
\]

(E5.1.7)

then using (5.3.11) and (5.3.26-27) the constitutive relation for an isotropic linearly elastic material can be written as

\[
\sigma_{ij} = 2\mu e_{ij} + K \varepsilon_{kk} \delta_{ij}
\]

(E5.1.8)

The strain energy (5.3.16) for an isotropic material is given by

\[
W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}
\]

\[
= \frac{1}{2} \left( s_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij} \right) \left( e_{ij} + \frac{1}{3} \varepsilon_{mm} \delta_{ij} \right)
\]

\[
= \mu e_{ij} e_{ij} + \frac{1}{2} K (\varepsilon_{kk})^2
\]

(E5.1.9)

Positive definiteness of the strain energy \( W \geq 0 \), equality iff \( \varepsilon = 0 \) imposes restrictions on the elastic moduli (see Malvern, for example). For the case of isotropic linear elasticity positive definitness of \( W \) requires

\[
K > 0 \quad \text{and} \quad \mu > 0 \quad \text{or} \quad E > 0 \quad \text{and} \quad -1 < \nu < \frac{1}{2}
\]

(E5.1.10)

Exercise 5.2. Derive these conditions by considering appropriate deformations. For example, to derive the condition on the shear modulus, \( \mu \), consider a purely deviatoric deformation and the positive definitness requirement.

Incompressibility.

The particular case of \( \nu = 1/2 \) (\( K = \infty \)) corresponds to an incompressible material. In an incompressible material in small deformations, the trace of the strain tensor must vanish,
i.e., $\varepsilon_{kk} = 0$. Deformations for which this constraint is observed are called isochoric. From (5.3.33) it can be seen that, for an incompressible material, the pressure cannot be determined from the constitutive relation. Rather, it is determined from the momentum equation. Thus, the constitutive relation for an incompressible, isotropic linear elastic material is written as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} \quad (5.3.26)$$

where the pressure $p = -\sigma_{kk}/3$ is unspecified and is determined as part of the solution.

**Plane Strain**

For plane problems, the stress-strain relation (5.3.21) can be even further simplified. In plane strain, $\varepsilon_{i3} = 0$, i.e., $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0$. In finite element coding, the standard Voigt notation used above is often modified to accommodate a reduction in dimension of the matrices. Letting $12 \rightarrow 3$, the stress-strain relation for plane strain is written as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12}
\end{bmatrix}
\]

and in addition

$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}) = \nu(\sigma_{11} + \sigma_{22}) \quad (5.3.28)$$

**Plane Stress**

For plane stress, $\sigma_{i3} = 0$. The condition $\sigma_{33} = 0$ gives the relation

$$\varepsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22}) = -\nu(\varepsilon_{11} + \varepsilon_{22}) \quad (5.3.29)$$

Letting $\kappa = 2\mu\lambda/(\lambda + 2\mu)$ and using (5.3.21), the stress-strain relation for plane stress is given by
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{12}
\end{bmatrix} =
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu \\
2(1-\nu)
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{12}
\end{bmatrix} \quad (5.3.30)
\]

**Axisymmetry**

For problems with an axis of symmetry (using a cylindrical polar coordinate system) the constitutive relation is given by

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{zz} \\
\sigma_{rz}
\end{bmatrix} =
\begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 \\
0 & 0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix}
e_{rr} \\
e_{\theta\theta} \\
e_{zz} \\
e_{rz}
\end{bmatrix} =
\begin{bmatrix}
1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\
\frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\
\frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix}
\begin{bmatrix}
e_{rr} \\
e_{\theta\theta} \\
e_{zz} \\
e_{rz}
\end{bmatrix} \quad (5.3.31)
\]

where

\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \quad (5.3.32)
\]

### 5.4. Multiaxial Nonlinear Elasticity

In this section, the small strain linear elasticity constitutive relations presented above will be extended to the case of finite strain. As will be seen, the extension to finite strains can be carried out in different ways and many different constitutive relations can be developed for multiaxial elasticity at large strains. In addition, because of the many different stress and deformation measures for finite strain, the same constitutive relation can be written in several different ways. It is important to distinguish between these two situations. The first case gives different material models while in the second, the same material model is represented by different mathematical expressions. In the latter, it is always possible to mathematically transform from one form of the constitutive relation to another.

The constitutive models for large strain elasticity are presented in order of increasing degree of what is commonly thought of as *elasticity*, i.e., hypoleasticity is presented first, followed by elasticity and finally hyperelasticity.

#### 5.4.1 Hypoelasticity

One of the simplest ways to represent elasticity at large strains, is to write the increments in stress as a function of the incremental deformation. As discussed in Section 3.7.2, in order to satisfy the principle of material frame indifference,
the stress increments (or stress rate) should be objective and should be related to an objective measure of the increment in deformation. A more detailed treatment of material frame indifference is given in the Appendix to this chapter and we will draw on that material as needed in the remainder of the chapter. Truesdell [ ] presented a general hypoelastic relation of the form

$$\nabla \sigma = f(\sigma, D)$$  \hspace{1cm} (5.4.1)

where $\sigma$ represents any objective rate of the Cauchy stress and $D$ is the rate of deformation tensor which is an objective tensor (see Equation (A.x)).

A large class of hypoelastic constitutive relations can be written in the form of a linear relation between the objective measure of stress and the rate of deformation tensor, i.e.,

$$\nabla \sigma = C : D$$  \hspace{1cm} (5.4.2)

In general, the fourth order tensor $C$ is a function of the stress state. As noted by Prager ( ), the relation (5.4.2) is rate-independent and incrementally linear and reversible. This means that for small increments about a finitely deformed state, the increments in stress and strain are linearly related and are recovered upon unloading. However, for large deformations, energy is not necessarily conserved and the work done in a closed deformation path is not-necessarily zero. It should be noted that the primary use of hypoelastic constitutive relations is in the representation of the elastic response in phenomenological elastic-plastic constitutive relations where the elastic deformations are small. In this case, dissipative effects are usually small also.

Some commonly used forms of hypoelastic constitutive relations are

$$\nabla_{J} \sigma = C' : D$$  \hspace{1cm} (5.4.3)

where $\sigma$ is the Jaumann rate of Cauchy stress given in equation (3.7.9) and

$$\mathcal{L}_v \tau = JC^T : D$$  \hspace{1cm} (5.4.4)

where $\mathcal{L}_v \tau$ is the Lie-Derivative of the Kirchhoff stress. Note that

$$\mathcal{L}_v \tau = \dot{\tau} - \tau \cdot L^T$$

$$= J(\dot{\sigma} - \sigma \cdot (\sigma \cdot L^T) + (\text{trace } L)\sigma)$$

$$= J \dot{\sigma}$$

$$\nabla_T \sigma$$

where $J = det F$ and $\sigma$ is the Truesdell rate of Cauchy stress. Thus the Lie-derivative of the Kirchhoff stress is simply the weighted Truesdell rate of the Cauchy stress. A more detailed discussion of Lie derivatives in the context of pull-back and push-forward operations
is given in the Appendix. We will use the concept of the Lie derivatives more extensively in our treatment of hyperelasticity (Section 5.4.3) and hyperelastic-plastic constitutive relations (Section 5.7.4).

Other forms of hypoelastic relations are based on the Green-Naghdi (also called the Dienes) rate which is denoted here by $\sigma$ and is given by

$$\nabla_\sigma \sigma = \dot{\sigma} - \dot{\Omega} \cdot \sigma - \sigma \cdot \Omega^T = R \cdot \frac{d}{dt} \left( R^T \cdot \sigma \cdot R \right) \cdot R^T$$

(5.4.6)

where

$$\Omega = \dot{R} \cdot R^T$$

(5.4.7)

is the spin associated with the rotation tensor $R$. The hypoelastic relation is given by

$$\nabla_\sigma \sigma = C^G : D$$

(5.4.8)

Note that the Green-Naghdi rate is a form of Lie Derivative (Appendix A.x) in that the Cauchy stress is pulled back by the rotation $R$ to the unrotated configuration where the material time derivative is taken with impunity and the result pushed forward by $R$ again to the current configuration. The quantity

$$\sigma = R^T \cdot \sigma \cdot R$$

(5.4.9)

is the co-rotational Cauchy stress (Equation 3.7.18) discussed in Chapter 3.

In the constitutive equations (5.4.3), (5.4.4) and (5.4.11) above, the fourth-order tensors of elastic moduli $C^J$, $C^T$ and $C^G$ are often taken to be constant and isotropic, e.g.,

$$C^J_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad C^J = \lambda I \otimes I + 2\mu I$$

(5.4.10)

Given a constitutive equation

$$\nabla_J \sigma = C^J : D$$

(5.4.11)

with constant moduli $C^J$ then, using the definition of the Jaumann stress rate (3.7.9) and the co-rotational rate (6.4.6), this relation can be written as

$$\nabla_R \sigma = C^J : D + (\Omega - W) \cdot \sigma + \sigma \cdot (\Omega - W)^T$$

(5.4.12)

which is a different constitutive equation to (5.4.8) with constant moduli $C^G$. 
5.4.2. Cauchy Elastic Material. As previously mentioned, an elastic material may be characterized as one which has no dependence on the history of the motion. The constitutive relation for a Cauchy elastic material is given by a special form of \((A.y)\) written as

$$\sigma = G(F)$$  \hfill (5.4.13)

where \(G\) is called the material response function and the explicit dependence on position \(X\) and time \(t\) has been suppressed for notational convenience. Applying the restriction \((A.z)\) due to material objectivity gives the form

$$\sigma = R \cdot G(U) \cdot R^T$$  \hfill (5.4.14)

Alternative forms of the same constitutive relation for other representations of stress and strain follow from the stress transformation relations in Box (3.2), e.g., the first Piola-Kirchhoff stress for a Cauchy elastic material is given by

$$P = J^{-1} \sigma \cdot F^{-T}$$

$$= J^{-1} R \cdot G(U) \cdot R^T \cdot R \cdot U^{-1}$$  \hfill (5.4.15)

while the relationship for the second Piola-Kirchhoff stress takes the form

$$S = J^{-1} F^{-1} \sigma \cdot F^{-T}$$

$$= J^{-1} U^{-1} \cdot R^T \cdot R \cdot G(U) \cdot R^T \cdot R \cdot U^{-1}$$

$$= J^{-1} U^{-1} \cdot R^T \cdot G(U) \cdot U^{-1} = h(U) = \tilde{h}(C)$$  \hfill (5.4.16)

where \(C = F^T \cdot F = U^2\) is the right Cauchy Green deformation tensor. For a given the motion, the deformation gradient is always known by its definition \(F = \partial x / \partial X\) (Equation 3.2.14). The stresses can therefore be computed for a Cauchy elastic material by (5.4.13) or one of the specialized forms (5.4.14-5.4.16) independent of the history of the deformation. However, the work done may depend on the deformation history or load path. Thus, while the material is history independent, it is in a sense \(path\) dependent. This apparent anomaly arises from the complications of large strain theory (see Example 5.1) below. In material models for small deformations, the work done in history-independent materials is always path-independent.

To account for material symmetry, we note that following Noll ( ) (see Appendix for further discussions of material symmetry) the stress field remains unchanged if the material is initially rotated by a rotation which belongs to the symmetry group of the material, i.e., if the deformation gradient, \(F\) is replaced by \(F \cdot Q\) where \(Q\) is an element of the symmetry group. Thus (5.4.13) is written as

$$\sigma = G(F \cdot Q)$$  \hfill (5.4.17)
For an initially isotropic material, all rotations belong to the symmetry group (5.4.17) must therefore hold for the special case $Q = R^T$, i.e.,

$$\sigma = G(F \cdot R^T) = G(V) \quad (5.4.18)$$

where the right polar decomposition (3.7.7) of the deformation gradient has been used.

It can be shown (Malvern, ) that for an initially isotropic material, the Cauchy stress for a Cauchy elastic material is given by

$$\sigma = \alpha_0 I + \alpha_1 V + \alpha_2 V^2 \quad (5.4.19)$$

where $\alpha_0$, $\alpha_1$, and $\alpha_2$ are functions of the scalar invariants of $V$. For further discussion of the invariants of a second order tensor, see Box 5.x below. The expression (5.4.19) is a special case of the general relation for an isotropic material given in (5.4.18).

**Example 5.1.** Consider a Cauchy Elastic material with constitutive relation given by

$$\sigma = \alpha(V - I), \quad \alpha = \alpha_0 J, \quad J = \det V \quad (E5.1.1)$$

Let the motion be given by

$$R = I, \quad F = V = \sum_{i=1}^{3} \lambda_i e_i \otimes e_i \quad (E5.1.2)$$

with $\lambda_3 = 1$ and $\lambda_1 = \lambda_1(t)$, $\lambda_2 = \lambda_2(t)$.

The principle stretches for two deformation paths 0AB and 0B are shown in Figure 5.y below:
Show that the work done in deforming the material along paths 0AB and 0B is different, i.e., path-dependent.

\[ V = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{bmatrix} \quad \sigma = \alpha \begin{bmatrix} \lambda_1 - 1 \\ \lambda_2 - 1 \\ 0 \end{bmatrix} \]  
\[ (E5.1.3) \]

Here,

\[ D = \dot{V}V^{-1} = \begin{bmatrix} \dot{\lambda}_1 / \lambda_1 \\ \dot{\lambda}_2 / \lambda_2 \\ 0 \end{bmatrix} \]  
\[ (E5.1.4) \]

\[ J = \text{det } V = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_2 \]  
\[ (E5.1.5) \]

The stress power is given by

\[ \dot{W} = \sigma : D \]

\[ = \alpha_0 \lambda_1 \lambda_2 (\lambda_1 - 1) \dot{\lambda}_1 / \lambda_1 + \alpha_0 \lambda_1 \lambda_2 (\lambda_2 - 1) \dot{\lambda}_2 / \lambda_2 \]  
\[ = \alpha_0 \lambda_2 (\lambda_1 - 1) \dot{\lambda}_1 + \alpha_0 \lambda_1 (\lambda_2 - 1) \dot{\lambda}_2 \]  
\[ (E5.1.6) \]

Path 0AB:
\[ dW = \alpha_0 \lambda_2 (\lambda_1 - 1) d\lambda_1 + \alpha_0 \lambda_1 (\lambda_2 - 1) d\lambda_2 \]  
(E5.1.7)

On OA, \( \lambda_2 = 1, \) constant. On AB \( \lambda_1 = \bar{\lambda}_1, \) constant. Thus

\[ W = \alpha_0 \bar{\lambda}_1 \bar{\lambda}_2 \left( \frac{\lambda_1}{2} - 1 \right) + \alpha_0 \bar{\lambda}_1 \bar{\lambda}_2 \left( \frac{\lambda_2}{2} - 1 \right) \]  
(E5.1.8)

Path 0AB:

\[ \lambda_2 = m \lambda_1 \quad m = \frac{\bar{\lambda}_2}{\bar{\lambda}_1} \]  
(E5.1.9)

\[ dW = \alpha m \lambda_1 (\lambda_1 - 1) d\lambda_1 + \frac{\alpha_1}{m} \lambda_2 (\lambda_2 - 1) d\lambda_2 \]  
(E5.1.10)

\[ W = \alpha m \left( \frac{\lambda_1^3}{3} - \frac{\lambda_1^2}{2} \right) + \frac{\alpha_1}{m} \left( \frac{\lambda_2^3}{3} - \frac{\lambda_2^2}{2} \right) d\lambda_2 \]  
(E5.1.11)

which differs from Eq. (E5.1.8), i.e., the work done is path-dependent.

**Exercise 5.2.** Show that the constitutive relation \( \sigma = \alpha_0 (V - I) \) gives a path-independent result for the two paths considered in Example 5.1 above.

Rate (or incremental) forms of the constitutive relation are required in the treatment of linearization (Chapter 6). A useful starting point for derivation of the rate form of the constitutive relation is, where possible, to take the material time derivative of the expression for the second Piola-Kirchhoff stress \( S \). Thus, for a Cauchy elastic material

\[ \dot{S} = \frac{\partial \tilde{h}(C)}{\partial C} : \dot{C} \]  
(5.4.20)

The fourth order tensor \( C^{SC} = \partial \tilde{h}(C)/\partial (C) \) is called the instantaneous tangent modulus. From the symmetries of \( S \) and \( C \), the tangent modulus possesses the minor symmetries, i.e., \( C^{SC}_{ijkl} = C^{SC}_{jikl} = C^{SC}_{ijlk} \).

**5.4.3. Hyperelastic Materials**

Elastic materials for which the work done on the material is independent of the load path are said to be hyperelastic (or Green elastic materials). In this section, some general features of hyperelastic materials are considered and then examples of hyperelastic constitutive models which are widely used in practice are given. Hyperelastic materials are characterized by the existence of a stored (or strain) energy function which is a potential for the stress. Note that from Eq. (5.4.16) the second Piola-Kirchhoff stress for a Cauchy elastic material can be written as
\[ \dot{S} = \tilde{h}(C) \]  

(5.4.21)

where \( C = F^T \cdot F = U^2 \) is the right Cauchy Green deformation tensor. For the case of a hyperelastic material, the second-order tensor \( \tilde{h} \) is derived from a potential, i.e.,

\[ S = \tilde{h}(C) = 2 \frac{\partial \Psi(C)}{\partial C} \]  

(5.4.22)

where \( \Psi \) is called the stored energy function. Expressions for different stress measures are obtained through the appropriate transformations (given in Box (3.2)), e.g.,

\[ \tau = J\sigma = F \cdot S \cdot F^T = 2F \cdot \frac{\partial \Psi(C)}{\partial C} \cdot F^T \]  

(5.4.23)

It can be shown (Marsden and Hughes) that, given (5.4.22), the Kirchhoff stress is also derivable from a potential, i.e.,

\[ \tau = 2 \frac{\partial \Psi(g)}{\partial g} \]  

(5.4.24)

where \( g \) is the spatial metric tensor (which is equivalent to the identity tensor for Euclidean spaces).

A consequence of the existence of a stored energy function is that the work done on a hyperelastic material is independent of the deformation path. This behavior is approximately observed in many rubber-like materials. To illustrate the independence of work on deformation path, consider the stored energy per unit reference volume in going from deformation state \( C_1 \) to \( C_2 \). Since the second Piola-Kirchhoff stress tensor \( S \) and the Green strain \( E = \frac{(C - I)}{2} \) are work conjugates,

\[ \frac{1}{2} \int_{C_1}^{C_2} S : dC = \Psi(C_1) - \Psi(C_2) \]  

(5.4.25)

which depends only on the initial and final states of deformation and is therefore independent of the deformation (or load) path. (Contrast this with the behavior of the Cauchy elastic material in Example 5.1 above.)

The rate forms of constitutive equations for hyperelastic materials and the corresponding moduli can be obtained by taking the material time derivative of Eq. (5.4.22) as follows:
\[ \dot{S} = \frac{\partial h(C)}{\partial C} : \dot{C} \]
\[ = 4 \frac{\partial^2 \Psi(C)}{\partial C \partial \dot{C}} : \dot{C} \]  
\[ = C^{SC} \frac{\dot{C}}{2} \]  

(5.4.26)

where

\[ C = 4 \frac{\partial^2 \Psi(C)}{\partial C \partial \dot{C}} \]  

(5.4.27)

is the tangent modulus. It follows that the tangent modulus for a hyperelastic material has the major symmetry \( C^{SC}_{ijkl} = C^{SC}_{klij} \) in addition to the minor symmetries shown already for the Cauchy elastic material.

It is often desirable (particularly in the linearization of the weak form of the governing equations (Chapter 6) to express the stress rate in terms of an Eulerian stress tensor such as the Kirchhoff stress. To this end we recall the Lie Derivative (also referred to as the convected rate) of the Kirchhoff stress introduced earlier in this Chapter, i.e.,

\[ L_v \tau = F \cdot \frac{d}{dt} \left( F^{-1} \cdot \tau \cdot F^{-T} \right) \cdot F^T = F \cdot \dot{S} \cdot F^T \]

\[ = \tau - L \cdot \tau - \tau \cdot L^T \]

\[ = \phi_x \left( \frac{d}{dt} \phi^*(\tau) \right) \]  

(5.4.28)

Note that the right Cauchy Green deformation tensor can be written as \( C = F^T \cdot F = F^T \cdot g \cdot F \) where \( g \) is the spatial metric tensor. In Euclidean space, we have \( g = I \) the identity tensor. Noting also that \( \dot{C}/2 = F^T \cdot D \cdot F \) it follows that the rate of deformation tensor can be written as

\[ D = F^{-T} \cdot \frac{d}{dt} \left( F^T \cdot g \cdot F \right) \cdot F^{-1} = \frac{1}{2} L_v g = \phi_x \left( \frac{d}{dt} \phi^*(\frac{g}{2}) \right) \]  

(5.4.29)

where \( L_v g \) is the Lie derivative of the spatial metric tensor. Using Eqs. (5.4.29) and (5.4.26) in (5.4.28) gives

\[ L_v \tau = C^{\tau D} : D \]  

(5.4.30)

where

\[ C^{\tau D}_{ijkl} = F_{im} F_{jn} F_{kp} F_{lq} C^{SC}_{mnpq} \]
are referred to as the spatial tangent moduli. It can be seen from the above that the Lie derivative of the Kirchhoff stress arises naturally as a stress rate in finite strain elasticity.

- Issues of uniqueness and stability of solutions in finite strain elasticity are mathematically complex. The reader is referred to [Ogden] and [Marsden and Hughes] for a detailed description.

It can be shown that, using the representation theorem (Malvern, 1969), the stored (strain) energy for a hyperelastic material which is isotropic with respect to the initial, unstressed configuration, can be written as a function of the principal invariants \( I_1, I_2, I_3 \) of the right Cauchy-Green deformation tensor, i.e., \( W = W(\mathbf{C}) \). The principal invariants of a second order tensor and their derivatives figure prominently in elastic and elastic-plastic constitutive relations. For reference, Box 5.1 summarizes key relations involving principal invariants.

---

**Box 5.1**

**Principal Invariants**

The principal invariants of a second order tensor \( \mathbf{A} \) are given by

\[
I_1(\mathbf{A}) = \text{Trace } \mathbf{A} \\
I_2(\mathbf{A}) = \frac{1}{2} \left\{ (\text{Trace } \mathbf{A})^2 - \text{Trace } \mathbf{A}^2 \right\} \\
I_3(\mathbf{A}) = \det \mathbf{A}
\]

(5.1.1)

When the tensor in question is clear from the context, the argument \( \mathbf{A} \) is omitted and the principal invariants denoted simply as \( I_1, I_2, I_3 \).

If \( \mathbf{A} \) is symmetric, then \( \mathbf{A} = \mathbf{A}^T \) and a set of 3 real eigenvalues (or principal values) of \( \mathbf{A} \) may be formed and written as \( \lambda_1, \lambda_2, \lambda_3 \). Then

\[
I_1 = \lambda_1 + \lambda_2 + \lambda_3 \\
I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\
I_3 = \lambda_1\lambda_2\lambda_3
\]

(B5.1.2)

The derivatives of the principal invariants of a second order tensor with respect to the tensor itself are often required in constitutive equations and in the linearization of the weak form (Chapter 6). For reference:

\[
\frac{\partial I_1}{\partial \mathbf{A}} = \mathbf{I}; \quad \frac{\partial I_1}{\partial \mathbf{A}_{ij}} = \delta_{ij} \\
\frac{\partial I_2}{\partial \mathbf{A}_{ij}} = I_1 \mathbf{I} - \mathbf{A}^T; \quad \frac{\partial I_2}{\partial \mathbf{A}_{ij}} = \mathbf{A}_{kk} \delta_{ij} - \mathbf{A}_{ji} \\
\frac{\partial I_3}{\partial \mathbf{A}_{ij}} = I_3 \mathbf{A}^{-T}; \quad \frac{\partial I_3}{\partial \mathbf{A}_{ij}} = I_3 \mathbf{A}_{ji}^{-1}
\]

(B5.1.3)

(B5.1.4)

(B5.1.5)

---

The second Piola-Kirchhoff stress tensor is given by ( ). Thus, for an isotropic material we have

\[
\mathbf{S} = 2 \frac{\partial w}{\partial \mathbf{C}} = 2 \left( \frac{\partial w}{\partial I_1} + I_1 \frac{\partial w}{\partial I_2} \right) \mathbf{I} - 2 \frac{\partial w}{\partial I_2} \mathbf{C} + 2I_3 \frac{\partial w}{I_3} \mathbf{C}^{-1}
\]

(5.1.32)
The Kirchhoff stress tensor is given by

\[ \tau = F \cdot S \cdot F^T = 2 \left( \frac{\partial w}{\partial I_1} + I_1 \frac{\partial w}{\partial I_2} \right) \mathbf{B} - 2 \frac{\partial w}{\partial I_2} \mathbf{B}^2 + 2I_3 \frac{\partial w}{I_3} \mathbf{I} \]

where \( \mathbf{B} = F \cdot F^T \) is the left Cauchy-Green deformation tensor. Note that \( S \) is co-axial has the same principal directions) with \( C \) while \( \tau \) is co-axial with \( B \). These results will be used below in deriving expressions for the stress tensors for specific hyperelastic models.

In the remainder of this section, examples of hyperelastic materials which are frequently used to model the behavior of rubber-like materials are presented.

**Neo-Hookean Material.** The stored energy function for a compressible Neo-Hookean material [Ref] (isotropic with respect to the initial, unstressed configuration) is written as

\[ w(C) = \frac{1}{2} \lambda_0 (\log J)^2 - \mu_0 \log J + \frac{1}{2} \mu_0 (\text{trace } C - 3) \]  

(5.4.34)

From Eq. (5.4.32), the stresses are given by

\[ S = \lambda_0 \log J \mathbf{C}^{-1} + \mu_0 (\mathbf{I} - \mathbf{C}^{-1}) \]

\[ \tau = \lambda_0 \log J \mathbf{I} + \mu_0 (\mathbf{B} - \mathbf{I}) \]  

(5.4.35)

Letting

\[ \lambda = \lambda_0, \quad \mu = \mu_0 - \lambda \log J \]  

(5.4.36)

and using Eqs. (5.4.27) and (5.4.31), the elasticity tensors (tangent moduli) are written in component form on \( \Omega_0 \) as

\[ C_{ijkl}^{SC} = \lambda C_{ij}^{-1} C_{kl}^{-1} + \mu \left( C_{ik}^{-1} C_{jl}^{-1} + C_{il}^{-1} C_{kj}^{-1} \right) \]  

(5.4.37)

and on \( \Omega \) as

\[ C_{ijkl}^{\tau D} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj} \right) \]  

(5.4.38)

The elasticity tensor in Eq. (5.4.38) has the same form as in Hooke's Law for small strain elasticity, except for the dependence of the shear modulus \( \mu \) on the deformation (see Eq. 5.4.36). Here \( \lambda_0 \) and \( \mu_0 \) are the Lamé constants of the linearized theory. Near incompressible behavior is obtained for \( \lambda_0 \gg \mu_0 \).

**Saint Venant - Kirchhoff Model.** A wide class of engineering problems can be studied by linear elastic material behavior. If the effects of large deformation are primarily due to rotations (such as in the bending of a marine riser or a fishing rod, for example) a straightforward generalization of Hooke's law to finite strains is often adequate. The Saint Venant-Kirchhoff model accomplishes this through the use of the Green strain measure \( E \) as follows. Let
\[ w(C) = W(E) = \frac{1}{2} E : C^{SE} \cdot E \quad (5.4.39) \]

where

\[ C_{ijkl}^{SE} = \lambda_0 \delta_{ij} \delta_{kl} + \mu_0 \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj} \right) \quad (5.4.40) \]

and where \( \lambda_0 \) and \( \mu_0 \) are Lamé constants. Noting that

\[ \mu_0 \]

and that

\[ \{\text{bf S}\} = 2 \{\partial \Psi(\{\text{bf C}\}) / \partial \{\text{bf C}\}\} = \{\partial W(\{\text{bf E}\}) / \partial \{\text{bf E}\}\} \]

the components \( S_{ij} \) of the second Piola-Kirchhoff stress are given by

\[ S_{ij} = \lambda_0 E_{kk} \delta_{ij} + 2\mu_0 E_{ij} \quad (5.4.41) \]

or

\[ \{\text{bf S}\} = \lambda_0 \text{trace}(\{\text{bf E}\}) \{\text{bf I}\} + 2\mu_0 \{\text{bf E}\} = \mathbf{D} : \{\text{bf E}\} \]

Because the Green strain tensor is symmetric, it follows that the stress tensor \( \{\text{bf S}\} \) is also symmetric. From Eq. (5.4.41) it is apparent that

fourth order material response tensor possesses major and minor symmetries. Because \( \{\text{bf E}\} \), \( \{\text{bf F}\} \)
and $\mathbf{C}$ are related ( ), it can also be shown that the components of
the nominal stress tensor is given by
\begin{equation}
\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}^T}, \quad P_{ij} = \frac{\partial W}{\partial F_{ji}}
\end{equation}

As the deformation gradient tensor $\mathbf{F}$ is not necessarily symmetric,
the 9 components
of the nominal stress tensor $\mathbf{P}$ do not necessarily possess symmetry.
Employing Eq. ( ), the Cauchy stress tensor $\mathbf{\sigma}$ is related to $W$ by:
\begin{equation}
\mathbf{\sigma} = \frac{1}{J} \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{F}^T} = \frac{1}{J} \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}} \cdot \mathbf{F}^T
\end{equation}
(Exercise: Show this.)

\indent \textbf{Modified Mooney-Rivlin Material}

In 1951, Rivlin and Saunders [ ] published their experimental results on the
large elastic deformations of vulcanized rubber - an incompressible
homogeneous isotropic elastic solid, in the Journal of Phil. Trans. A.,
Vol. 243, pp. 251-288. This material model with a few refinements is
still the most commonly used model for rubber materials. It is assumed
in the model that behavior of the material is initially isotropic and
path-independent, i.e., a stored energy function exists. The stored
energy function is written
\begin{equation}
\Psi = \Psi(\{\text{bf } C\}) = W(I_1, I_2, I_3)
\end{equation}
where $I_1$, $I_2$ and $I_3$ are the three scalar invariants of $\{\text{bf } C\}$.
Rivlin and Saunders considered an initially isotropic nonlinear
elastic incompressible material ($I_3=1$) then
\begin{equation}
\Psi = \Psi(I_1, I_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \bar{c}_{ij} (I_1-3)^i (I_2-3)^j, \quad \bar{c}_{00}=0
\end{equation}
where $\bar{c}_{ij}$ are constants.
They performed a number of experiments on different
types of rubbers and discovered that Eq. ( ) may be reduced to
\begin{equation}
\Psi = c(I_1-3) + f(I_2-3)
\end{equation}
where $c$ is a constant and
$f$ is a function of $I_2 - 3$. For a Mooney-Rivlin material, $W$ can be
reduced further to
\begin{equation}
\Psi = \Psi(I_1, I_2) = c_1 (I_1 - 3) + c_2 (I_2 - 3)
\end{equation}
An example of the set of $c_1$ and $c_2$ is: $c_1 = 18.35 \text{ psi}$ and $c_2 = 1.468 \text{ psi}$. Equation (1) is also an example of a Neo-Hookean material, and the components of the second Piola-Kirchhoff stress can be obtained by differentiating Eq. (1) with respect to the components of the right Cauchy Green deformation tensor; however, the deformation is constrained such that

$$\begin{equation}
\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}}, \quad \text{with} \quad I_3 = \det \mathbf{C} = 1
\end{equation}$$

The condition $I_3 = 1$ simply implies that $J=1$ and there is no volume change. The condition can be written as

$$\begin{equation}
\text{ln}I_3 = 0
\end{equation}$$

which represents a constraint on the deformation. One way in which the constraint (1) can be enforced is through the use of a constrained potential, or stored energy, function [Ref]. Alternatively, a penalty function formulation (Hughes, 1987) can be used. In this case, the modified strain energy function and the constitutive equation become:

$$\begin{eqnarray}
\bar{\Psi} &=& \Psi + p_0 \text{ln}I_3 + \frac{1}{2} \lambda (\text{ln}I_3)^2 \\
\mathbf{S} &=& 2\frac{\partial \Psi}{\partial \mathbf{C}} + 2(p_0 + \lambda (\text{ln}I_3))^2 \\
\end{eqnarray}$$
respectively. The penalty parameter $\lambda$ must be large enough so that the compressibility error is negligible (i.e., $I_3$ is approximately equal to $1$), yet not so large that numerical ill-conditioning occurs. Numerical experiments reveal that $\lambda = 10^3 \times \max(C_1, C_2)$ to $\lambda = 10^7 \times \max(C_1, C_2)$ is adequate for floating-point word length of 64 bits. The constant $p_o$ is chosen so that the components of $\bf S$ are all zero in the initial configuration, i.e.,

$$p_o = -(C_1 + 2C_2)$$

$\bullet$ Exercises

\setcounter{equation}{0}

\subsection{Plasticity in One Dimension}

Materials for which permanent strains are developed upon unloading are called plastic materials. Many materials (such as metals) exhibit elastic (often linear) behavior up to a well defined stress level called the yield strength. Once loaded beyond the initial yield strength, plastic strains are developed. Elastic plastic materials are further subdivided into rate-independent materials, where the stress is independent of the strain rate, i.e., the rate of loading has no effect on the stresses and rate-dependent materials, in which the stress depends on the strain
rate; such materials are often called strain rate-sensitive.

The major ingredients of the theory of plasticity are

\begin{enumerate}
\item A decomposition of each increment of strain into an elastic, reversible component $d\varepsilon^e$ and an irreversible plastic part $d\varepsilon^p$.

\item A yield function $f$ which governs the onset and continuance of plastic deformation.

\item A flow rule which governs the plastic flow, i.e., determines the plastic strain increments.

\item A hardening relation which governs the evolution of the yield function.
\end{enumerate}

There are two classes of elastic-plastic laws:

\begin{itemize}
\item Associative models, where the yield function and the potential function are identical

\item Nonassociative models where the yield function and flow potential are different.
\end{itemize}
Elastic-plastic laws are path-dependent and dissipative. A large part of the work expended in deforming the solid is irreversibly converted to other forms of energy, particularly heat, which can not be converted to mechanical work. The stress depends on the entire history of the deformation, and can not be written as a single valued function of the strain as in ( ) and ( ). The stress is path-dependent and dependes on the history of the deformation. We cannot therefore write an explicit relation for the stress in terms of strain, but only as a relation between rates of stress and strain.

The constitutive relations for rate-independent and rate-dependent plasticity in one-dimension are given in the following sections.

\textbf{Rate-Independent Plasticity in One-Dimension}

A typical stress-strain curve for a metal under uniaxial stress is shown in Figure~\ref{fig:stress-strain}. Upon initial loading, the material behaves elastically (usually assumed linear) until the initial yield stress is attained. The elastic regime is followed by an elastic-plastic regime where permanent irreversible plastic strains are induced upon further loading. Reversal of the stress is called unloading. In unloading, the stress-strain response is typically assumed to be governed by the elastic modulus and the strains which remain after complete unloading are called
the plastic strains. The increments in strain are assumed to be additively decomposed into elastic and plastic parts. Thus we write

\begin{equation}
d\varepsilon = d\varepsilon^e + d\varepsilon^p
\end{equation}

Dividing both sides of this equation by a differential time increment $dt$ gives the rate form

\begin{equation}
\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p
\end{equation}

The stress increment (rate) is related to the increment (rate) of elastic strain. Thus

\begin{equation}
d\sigma = E d\varepsilon^e, \quad \dot{\sigma} = E \dot{\varepsilon}^e
\end{equation}

relates the increment in stress to the increment in elastic strain.

In the nonlinear elastic-plastic regime, the stress-strain relation is given by

( see Figure ( ) )

\begin{equation}
d\sigma = Ed\varepsilon^e = E^\text{tan} d\varepsilon
\end{equation}

where the elastic-plastic tangent modulus, $E^\text{tan}$, is the slope of the
stress-strain curve. In rate form, the relation is written as
\begin{equation}
\dot{\sigma} = E\dot{\varepsilon^e} = E^\text{\rm tan}\dot{\varepsilon}
\end{equation}

The above relations are homogeneous in the rates of stress and strain which means that if time is scaled by an arbitrary factor, the constitutive relation remains unchanged and therefore the material response is rate-independent even though it is expressed in terms of a strain rate. In the sequel, the rate form of the constitutive relations will be used as the notation because the incremental form can get cumbersome especially for large strain formulations.

$\bullet$ kinematic hardening

The increase of stress after initial yield is called work or strain hardening. The hardening behavior of the material is generally a function of the prior history of plastic deformation. In metal plasticity, the history of plastic deformation is often characterized by a single quantity $\bar{\varepsilon}$ called the accumulated plastic strain which is given by
\begin{equation}
\bar{\varepsilon} = \int \dot{\bar{\varepsilon}} \text{d}t
\end{equation}
where
\begin{equation}
\bar{\varepsilon} = \int \dot{\bar{\varepsilon}} \text{d}t
\end{equation}
\[ \dot{\bar{\varepsilon}} = \sqrt{\dot{\varepsilon}^p \dot{\varepsilon}^p} \]

is the effective plastic strain rate. The plastic strain rate is given by
\[ \dot{\varepsilon}^p = \dot{\lambda} \text{sgn}(\sigma) \]
where
\[ \text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0 \\ -1 & \text{if } \sigma < 0 \end{cases} \]
From ( ) it follows that
\[ \dot{\lambda} = \dot{\bar{\varepsilon}} \]
The accumulated plastic strain \( \bar{\varepsilon} \), is an example of an internal variable used to characterize the inelastic response of the material. An alternative, internal variable used in the representation of hardening is the plastic work which is given by (Hill, 1958)
\[ W^P = \int \sigma \dot{\varepsilon}^p \, dt \]
The hardening behavior is often expressed through the dependence of the yield stress, $Y$, on the accumulated plastic strain, i.e., $Y = Y(\bar{\varepsilon})$. More general constitutive relations use additional internal variables.

A typical hardening curve is shown in Figure ( ). The slope of this curve is the plastic modulus, $H$, i.e.,

\begin{equation}
H = \frac{dY(\bar{\varepsilon})}{d\bar{\varepsilon}}
\end{equation}

The effective stress is defined as

\begin{equation}
\bar{\sigma} = \sqrt{\sigma^2} \equiv |\sigma| = \sigma \text{ sgn}(\sigma)
\end{equation}

The yield condition is written as

\begin{equation}
f = \bar{\sigma} - Y(\bar{\varepsilon}) = 0
\end{equation}

which is regarded as the equation for the yield point (or surface when multiaxial stress states are considered). Note that the plastic strain rate can be written as

\begin{equation}
\dot{\varepsilon}^p = \dot{\bar{\varepsilon}} \text{ sgn}(\sigma) = \dot{\bar{\varepsilon}} \frac{\partial f}{\partial \sigma}
\end{equation}
where the result $\partial \bar{\sigma}/\partial \sigma = \text{sign}(\sigma)$ has been used. For plasticity in one-dimension (uniaxial stress),
the distinction between associated and non-associated plasticity is not possible. Also, the lateral strain which accompanies the axial strain has both elastic and plastic parts. This point will be addressed further in Section X on multiaxial plasticity.

Plastic deformation occurs only when the yield condition is met. Upon plastic loading, the stress must remain at yield, which is called the \textit{consistency} condition, and is given by
\begin{equation}
\dot{f} = \dot{\bar{\sigma}} - \dot{Y}(\bar{\varepsilon}) = 0.
\end{equation}

>From ( ) it follows that, during plastic loading,
\begin{equation}
\dot{\bar{\sigma}} = \frac{dY(\bar{\varepsilon})}{d\bar{\varepsilon}} \dot{\bar{\varepsilon}} = H \dot{\bar{\varepsilon}}
\end{equation}
Using ( ), ( ) and ( ) in ( ) gives
\begin{equation}
\frac{1}{E^\text{tan}} = \frac{1}{E} + \frac{1}{H}
\end{equation}
or
\begin{equation}
E^\text{tan} = \frac{EH}{E + H} = E - \frac{E^2}{E + H}
\end{equation}
The plastic switch parameter $\alpha$ is introduced with $\alpha=1$ corresponding to plastic loading and $\alpha=0$ corresponding to purely elastic response (loading or unloading). Thus the tangent modulus is written

\begin{equation}
E^{\text{tan}} = E - \alpha \frac{E^2}{E + H}
\end{equation}

An alternative way of writing the loading-unloading conditions without using the switch parameter $\alpha$ is through the use of the Kuhn-Tucker conditions, which play an important role in mathematical programming theory [Ref?]

For plasticity, the conditions are:

\begin{equation}
\dot{\lambda} \dot{f} = 0, \quad \dot{\lambda} \geq 0, \quad \dot{f} \leq 0
\end{equation}

Thus for plastic loading, $\dot{\lambda} \geq 0$ and the consistency condition $\dot{f} = 0$ is satisfied. For purely elastic loading or unloading, $\dot{f} \neq 0$ and it follows that $\dot{\lambda} = 0$.

The constitutive relations for rate-independent plasticity in 1D are summarized in Box 9.1.

\subsection*{Rate-Dependent Plasticity in One Dimension}
In rate dependent plasticity, the plastic response of the material depends on the rate of loading. The elastic response is given as before (in rate form) as
\begin{equation}
\dot{\sigma} = E\dot{\varepsilon}^e
\end{equation}
which may be written using the elastic-plastic decomposition of the total strain rate (Equation ) as
\begin{equation}
\dot{\sigma}=E(\dot{\varepsilon}-\dot{\varepsilon}^p).
\end{equation}
For plastic deformation to occur the yield condition must be met or exceeded. This differs from the rate-independent case in that in rate-dependent plasticity the stress can exceed the yield stress.
The plastic strain rate is given by
\begin{equation}
\dot{\varepsilon}^p = \dot{\bar{\varepsilon}} \text{sgn}\sigma
\end{equation}
For many rate-dependent materials, the plastic response is characterized by an overstress model of the form
\begin{equation}
\dot{\bar{\varepsilon}} = \frac{\phi (\sigma, \bar{\varepsilon})}{\eta}
\end{equation}
where, $\phi$ is the overstress and $\eta$ is the viscosity.
For example, the overstress model introduced by Perzyna (19xx) is given by
\begin{equation}
\phi = Y\bigl(\bar{\sigma}/Y - 1\bigr)^n
\end{equation}
where $n$ is called the rate-sensitivity exponent. Using ( ) and ( ) the expression for the stress rate is given by
\begin{equation}
\dot{\sigma} = E\biggl(\dot{\varepsilon} - \phi(\sigma, \bar{\varepsilon})/\eta \times \text{sgn}(\sigma)\biggr).
\end{equation}
which is a differential equation for the evolution of the stress. Comparing this expression to ( ), it can be seen that ( ) is inhomogeneous in the rates and therefore the material response is rate-dependent. Models of this type are often used to model the strain-rate dependence observed in materials. More elaborate models with additional internal variables and perhaps different response in different strain-rate regimes have been developed (see for example the unified creep plasticity model [Ref]). Nevertheless, the simple overstress model ( ) has been very successful in capturing the strain rate dependence of metals over a large range of strain rates [Refs].

An alternative form of rate-dependent plasticity that has been used with considerable success by Needleman ( ) and others is given by
\begin{equation}
\dot{\bar{\varepsilon}}^p = \dot{\varepsilon}_0\biggl(\bar{\sigma}/Y\biggr)^{1/m}
\end{equation}
without any explicit yield surface. For plastic straining at the rate $\dot{\varepsilon}_0$, the response $\bar{\sigma} = Y$ is obtained. This response is called the reference response and can be obtained by performing a uniaxial stress test with a plastic strain rate $\dot{\varepsilon}_0$. In the case of small elastic strain rates, the test can be run at total strain rate of $\dot{\varepsilon}_0$ without significant error (Check!). For rates which exceed $\dot{\varepsilon}_0$ the stress is elevated above the reference stress while for lower rates the stress falls below this value. A case of particular interest is the near rate-independent limit when the rate-sensitivity exponent $m \to 0$.

It can be seen from ( ) that, for $\bar{\sigma} < Y$, the effective plastic strain rate is negligible while for a finite plastic strain rate the effective stress is approximately equal to the reference stress, $Y$. In this way, the model exhibits an effective yield limit together with near elastic unloading and rate-independent response.

The constitutive relations for rate-dependent plasticity in 1D are summarized in Box 9.2